UNRESTRICTED SOLUTION FIELDS OF ALMOST-SEPARABLE DIFFERENTIAL EQUATIONS

BY

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PART I

1. The type in question is that particular case of a real, scalar differential equation

$$x' = F(x, t), \qquad (x' = dx/dt), \tag{1}$$

in which $F(x, t)$ is a function of x alone plus a function of t alone:

$$x' = g(x) + f(t). \tag{2}$$

Whereas the solution of (2) requires just a quadrature or the inversion of quadratures according as g or f vanishes identically, the general case of (2) is not solvable "explicitly." This negation follows from irreducibility theorems centering about Liouville's result on his case of Ricatti's equation.

There are, however, a few general facts, substantially *Tauberian in nature*, concerning the *asymptotic* integration of (2) when $f(t)$ becomes "small" as $t \to \infty$. The isolation of such situations is the purpose of the present paper. It turns out that a whole class of known theorems can be reduced to the asymptotic behavior of the solutions of (2). Correspondingly, these particular cases become subsumed to a unified and, at the same time, simplified treatment.

All coefficient functions and solutions will be meant to be real-valued.

2. One of the general theorems of the type in question can be formulated as follows:

(I) *Let $g(x)$ and $f(t)$ be continuous on the line $-\infty < x < \infty$ and on the half-line $0 \leqq t < \infty$, respectively. For $g(x)$, suppose that*

$$g(x) < 0 \quad \text{if} \quad 0 < x < \infty \quad \text{and} \quad g(x) > 0 \quad \text{if} \quad -\infty < x < 0 \tag{3}$$

and that

$$|g(x)| \to \infty \quad \text{as} \quad |x| \to \infty. \tag{4}$$

For $f(t)$, suppose that either

$$f(t) \to 0 \quad \text{as} \quad t \to \infty \tag{5'}$$

or

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$$\int_0^\infty f(t)dt = \lim_{T\to\infty} \int_0^T f(t)dt \textit{ converges} \tag{5''}$$

(possibly just conditionally); or, what requires less than either (5′) *or* (5″), *that*

$$\operatorname*{l.u.b.}_{0<v<\infty} \left| \int_u^{u+v} f(t)dt \right| \Big/ (1+v) \to 0 \quad \textit{as} \quad u \to \infty. \tag{5}$$

Then, as t increases, no continuation of any solution $x = x(t)$ of the differential equation (2) *can cease to exist at a finite value of t, and*

$$x(t) \to 0 \quad \textit{as} \quad t \to \infty \tag{6}$$

holds for every continuation of every solution of $x = x(t)$ of (2).

The formulation of the assertions in terms of "continuations" is necessary because no local restriction of the Lipschitz type is imposed on $g(x)$; so that more than one solution path of (2) can pass through a point of the (x, t)-plane.

The content of the first assertion of the theorem, that is, of the assertion preceding (6), can be illustrated by the example $g(x) = x^2$, $f(t) \equiv 0$. In this case, (2) reduces to $x' = x^2$, and so all solutions $x(t) \not\equiv 0$ are of the form $x = (c-t)^{-1}$, where c is arbitrary. If $c > 0$, this solution exists for $0 \leqq t < c$ but ceases to exist at a finite value of t, namely, at $t = c$. Thus the first assertion of the theorem is false in this case. Correspondingly, (3) is now violated.

If $x' = x^2$ is replaced by the case $g(x) = -x^3$, $f(t) = 0$ of (2), that is, by $x' = x^3$, then all conditions of the theorem are satisfied, and every solution $x(t) \not\equiv 0$ is of the form $x = (2t - c)^{-1/2}$. This solution exists for all large positive t, but when continued for decreasing t, it ceases to exist at a finite value of t, namely, at $t = 2c$. Hence, the specification following (5), namely, that t should *increase*, is essential indeed.

In (I), the condition (5) on $f(t)$ is necessary as well as sufficient. In fact, whenever $g(x)$, as in (I), is a continuous function vanishing at $x = 0$, the existence of *some* solution $x = x(t)$ satisfying (6) implies (5). For, a quadrature of (2) gives

$$\left| \int_u^{u+v} f(t)dt \right| \leqq | x(u+v) - x(u) | + \int_u^{u+v} | g(x(t)) | \, dt,$$

so that (6) and $g(x(t)) \to 0$, as $t \to \infty$, imply (5).

In the particular case (5′), but not in the particular case (5″), of (5), the theorem can be obtained from theorem (I) in [2](1), if $t = \infty$ is transformed to $t = +0$ by the substitution $t \to 1/t$. Correspondingly, the general theorems to be proved will depend on an adaptation of the considerations used in [6] and further developed in [2].

3. First, that assertion of (I) which precedes (6), that is, the existence of

(1) Numbers in brackets refer to the bibliography at the end of the paper.

indefinite continuations, will be verified.

It was proved in [4] that, if $F(x, t)$ is any real-valued, continuous function in the half-plane $t>0$ and if a solution $x=x(t)$ of (1) is given on an interval $t_0 \leqq t < t^0$, then this solution can be continued through and beyond $t=t^0$ unless

$$\text{(7)} \qquad \text{either } x(t) \to +\infty \quad \text{or} \quad x(t) \to -\infty \quad \text{as} \quad t \to t^0 - 0.$$

Hence, it is sufficient to show that, if (1) has the separated structure of (2), where $g(x)$ satisfies (3) and (4) (the restriction on $f(t)$, namely, assumption (5), is not needed at this stage), there cannot exist a finite t^0 satisfying (7).

Suppose, if possible, that there exists such a t^0. Then, since $f(t)$ is continuous for $0 \leqq t < \infty$, hence, bounded for $t_0 \leqq t \leqq t^0$, it is seen from (7), (3), (4), and (2) that

$$\text{(8)} \qquad \text{either } x'(t) \to -\infty \quad \text{or} \quad x'(t) \to +\infty \quad \text{as} \quad t \to t^0 - 0.$$

It is understood that the first and second alternatives are simultaneous in (7) and (8). Clearly, the respective cases of (8) contradict those of (7).

4. Accordingly, only (6) remains to be proved. To this end, it will first be shown that

$$\text{(9)} \qquad \liminf_{t\to\infty} x(t) \leqq 0 \leqq \limsup_{t\to\infty} x(t),$$

where $x(t)$ is any continuation of any solution of (2). This proof will involve assumptions (5), (3) and part of the assumption (4), namely, $\liminf_{x\to\infty} |g(x)| \neq 0$; that is,

$$\text{(10)} \qquad g^*(r) > 0 \quad \text{for} \quad 0 < r < \infty,$$

where

$$\text{(11)} \qquad g^*(r) = \operatorname*{g.l.b.}_{r<|x|<\infty} |g(x)|.$$

Suppose, if possible, that the first of the inequalities (9) is false, that is, that the lower limit of $x(t)$ as $t\to\infty$ is positive (possibly $+\infty$). Then there exist an $a>0$ and a $T>0$ such that

$$\text{(12)} \qquad x(t) > a \quad \text{if} \quad t > T.$$

It follows from (10) that, since $a>0$, it is possible to choose an ϵ satisfying

$$\text{(13)} \qquad 0 < \epsilon < g^*(a).$$

Since T in (12) can be replaced by any $T'>T$ if a is fixed, it can, in view of (5), be assumed that

$$\text{(14)} \qquad \left|\int_u^{u+v} f(t)dt\right| < \epsilon(1+v) \quad \text{if} \quad T < u < u+v < \infty,$$

when $T=T_\epsilon$.

According to (3), (12) and (11),

$$g(x(t)) \leqq -g^*(a) \quad \text{if} \quad t > T.$$

It follows therefore, from (14), that a quadrature of both members of (2) leads to

$$x(T+v) - x(T) < -g^*(a)v + \epsilon(1+v) \quad \text{if} \quad v > 0.$$

On the other hand, from (13),

$$-g^*(a)v + \epsilon(1+v) \to -\infty \quad \text{as} \quad v \to \infty.$$

The last two formula lines imply that

$$x(T+v) \to -\infty \quad \text{as} \quad v \to \infty,$$

where T is fixed. This contradicts (12).

The contradiction proves the first of the inequalities (9). The second of these inequalities can, of course, be proved in the same way.

5. It is seen from (9) that, in order to complete the proof of (6), it is sufficient to show that

$$\liminf_{t\to\infty} x(t) \geqq 0 \quad \text{and} \quad \limsup_{t\to\infty} x(t) \leqq 0. \tag{15}$$

The proof of (15) will be such as to make it clear that it is sufficient to consider only the second of the inequalities (15).

Suppose, if possible, that the latter is false. Then there exists a positive constant b such that $x(t)$ exceeds b for certain, arbitrarily large, values of t. On the other hand, the first of the inequalities (9) implies that $b/2$ exceeds $x(t)$ for some other, arbitrarily large, values of t. It follows, therefore, from the continuity of $x(t)$, that there exist certain, arbitrarily large, values of u and certain, corresponding, values $v=v_u>0$ satisfying the equations

$$x(u) = b/2, \qquad x(u+v) = b \tag{16}$$

and having the property that

$$x(t) \geqq b/2 \quad \text{if} \quad u \leqq t \leqq u+v.$$

The latter property implies, in view of (3) and (11), that

$$g(x(t)) \leqq -g^*(b/2) \quad \text{if} \quad u < t < u+v. \tag{17}$$

Since $b>0$, it follows from (10) that it is possible to choose an ϵ satisfying

$$0 < \epsilon < \min\,(g^*(b/2), b/2). \tag{18}$$

With reference to such an ϵ, let $T=T_\epsilon$ be chosen in accordance with (14). Then, if $u+v>u>T$, a quadrature of (2) gives

$$x(u+v)-x(u)<\int_u^{u+v} g(x(t))dt+\epsilon(1+v).$$

But there exist arbitrarily distant t-intervals, $u<t<u+v$, on which both (16) and (17) are satisfied. If this is combined with the last formula line, it follows that

$$b-b/2<-g^*(b/2)v+\epsilon(1+v).$$

Since the latter inequality can be written in the form

$$b/2+g^*(b/2)v<\epsilon(1+v), \qquad \text{where } b>0,\ v>0,$$

it contradicts (18). This contradiction completes the proof of (6) and, therefore, that of (I).

6. The following dual of (I) will now be proved:

(II) *Let $g(x)$ and $f(t)$ be continuous on the line $-\infty<x<\infty$ and the half-line $0\leqq t<\infty$, respectively. For $g(x)$, suppose* (4) *and the following dual of* (3):

$$(19) \qquad g(x)>0 \quad \text{if} \quad 0<x<\infty \quad \text{and} \quad g(x)<0 \quad \text{if} \quad -\infty<x<0.$$

For $f(t)$, suppose (5) *holds; for instance, either* (5′) *or* (5″). *Then* (2) *has at least one solution $x=x(t)$ which exists for all large t and satisfies* (6).

In contrast to the case of (I), the solution $x=x(t)$ of (2) satisfying (6) can be unique in the case (II). In fact, this will be the case if $g(x)$ is smooth in an appropriate sense. This is illustrated by the following.

REMARK. *If the assumptions of* (II) *are satisfied and if $g(x)$ is nondecreasing at every point of the x-line, then* (2) *has just one solution $x=x(t)$ satisfying* (6).

The proof of this remark will be omitted, since it will be clear from the proof of (II) itself that the proof of the remark is substantially identical with the uniqueness proof given in [2, pp. 306–307].

7. The proof of (II) proceeds as follows:

For every positive integer n, let $x=x_n(t)$ be a solution of (2) satisfying

$$x_n(n)=0.$$

Such a solution can be continued, with decreasing t, so as to exist for the interval $0\leqq t\leqq n$. For suppose the contrary. Then, corresponding to the result of [4] used in (7), there ought to exist on the interval $0<t<n$ a certain $t=t_n$ having the property that the solution $x_n(t)$ exists for $t_n<t\leqq n$ but

$$(20) \qquad \text{either } x_n(t)\to+\infty \quad \text{or} \quad x_n(t)\to-\infty \quad \text{as} \quad t\to t_n+0.$$

This leads via (19) to the same contradiction as (7) and (8) did via (3), the dual of (19).

This proves that $x_n(t)$ exists for $0\leqq t\leqq n$. It will now be shown that

$$x_n(t_n) \to 0 \qquad \text{as } n \to \infty \tag{21}$$

holds for every sequence $t_1, t_2, \cdots$ satisfying

$$0 < t_n < n \quad \text{and} \quad t_n \to \infty \qquad \text{as } n \to \infty. \tag{22}$$

Suppose, if possible, that (22) does not imply (21). Then there exists a sequence $t_2, t_2, \cdots$ satisfying (22) and containing a subsequence such that, if the nth element of this subsequence and the corresponding element of the sequence $x_1(t), x_2(t), \cdots$ are simply denoted by t_n and $x_n(t)$, respectively, then (22) is satisfied but

$$\text{either } x_n(t_n) > b \quad \text{or} \quad x_n(t_n) < -b \tag{23}$$

holds for every n and for a fixed $b>0$.

Consider the first of these two possibilities. Then, since $x_n(t)$ is a continuous function which vanishes at $t=n$, and since $t_n<n$, there belongs to every n a positive v_n satisfying

$$x_n(t_n + v_n) = b/2 \quad \text{and} \quad x_n(t) > b/2, \quad \text{if} \quad t_n < t < t_n + v_n. \tag{24}$$

If $g^*(r)$ is defined by (11), then (24) and (19) imply that

$$g(x_n(t)) \geqq g^*(b/2) \quad \text{if} \quad t_n < t < t_n + v_n. \tag{25}$$

Since (4), (19) and the continuity of $g(x)$ insure (10), and since $b>0$, there exists an ϵ satisfying (18). With reference to such an ϵ, let $T=T_\epsilon$ be chosen in accordance with (14). Then, if n is so large that $t_n>T$, a quadrature of (2) gives

$$x_n(t_n + v_n) - x_n(t_n) > \int_{t_n}^{t_n+v_n} g(x_n(t))dt - \epsilon(1 + v_n).$$

Hence, by (25) and the first of the relations (24),

$$b/2 - x_n(t_n) > g^*(b/2)v_n - \epsilon(1 + v_n),$$

and so, since the first of the two cases (23) is being considered,

$$\epsilon(1 + v_n) > g^*(b/2)v_n + b/2, \quad \text{where} \quad b > 0,\ v_n > 0.$$

But this contradicts (18).

Accordingly, the first of the possibilities (23) cannot occur. The second case of (23) can, of course, be ruled out in the same way. This proves that (21) holds for every sequence $t_1, t_2, \cdots$ satisfying (22).

8. The solution $x=x_n(t)$, which exists on the interval $0 \leqq t \leqq n$ (at least), is such that

$$|x_n(t)| < C \qquad \text{if } 0 \leqq t \leqq n, \tag{26}$$

where C is a bound independent of t and n. For, if there did not exist such a C, it would be possible to construct a sequence $t_1, t_2, \cdots$ satisfying (22) but violating (21).

If N is arbitrarily fixed, then, since $g(x)$ and $f(t)$ are continuous throughout, it is seen from (2) and (26) that the functions $x_N(t)$, $x_{N+1}(t), \cdots$, besides being uniformly bounded, have uniformly bounded derivatives on the interval $0 \leqq t \leqq N$ and are, therefore, equicontinuous there. Hence, the compactness of such functions, when followed by the diagonal process corresponding to $N \rightarrow \infty$ in $0 \leqq t \leqq N$, shows that the sequence $x_1(t)$, $x_2(t), \cdots$ contains a subsequence which tends to a limit function uniformly on every fixed bounded t-interval of the half-line $0 \leqq t < \infty$.

If $x = x(t)$ denotes this limit function, then, since every $x_n(t)$ is a solution of (2) (when $0 \leqq t \leqq n$), and since $g(x)$ and $f(t)$ are continuous throughout, it is clear from the local uniform convergence of the selected subsequence that $x = x(t)$ is a solution of (2). Since this solution exists for $0 \leqq t < \infty$, it follows that the proof of (II) will be complete if it is shown that this solution satisfies (6).

Let the nth element of that subsequence of $x_1(t)$, $x_2(t), \cdots$ which tends to $x(t)$ be denoted simply by $x_n(t)$. Thus

$$x_n(t) \rightarrow x(t) \qquad (n \rightarrow \infty)$$

holds uniformly on every fixed bounded interval of the half-line. Hence, it is clear that the negation of (6) leads to a sequence $t_1, t_2, \cdots$ which satisfies (22) but violates (21). Since such a sequence $t_1, t_2, \cdots$ cannot exist, the proof of (II) is complete.

9. Let a solution $x = x(t)$ of (2) be called an *unrestricted solution* if it does not cease to exist when t increases indefinitely, that is, if (7) does not materialize at any $t^0 < \infty$. For instance, all solutions of (2) are unrestricted under the assumptions of (I) and at least one solution is unrestricted under the assumptions of (II). In the latter regard, additional restrictions on $g(x)$ lead to the following refinement of (II):

(II*) *Let* $g(x)$ *and* $f(t)$ *satisfy the assumptions of* (II). *In addition, suppose that*

$$\int_{-\infty} |g(x)|^{-1} dx + \int^{\infty} |g(x)|^{-1} dx < \infty, \tag{27}$$

and that $g(x)$ *is such as to make the function*

$$g(x + x_1)/g(x) \tag{28}$$

of the two independent variables x, x_1 *bounded as* $(|x|, |x_1|) \rightarrow (\infty, 0)$. *Then* (6) *holds for every unrestricted solution of* (2).

That the additional restrictions, (27) and (28), cannot be omitted is shown by the example $x' = x$. Here $g(x) = x$ and $f(t) = 0$. Hence, all assumptions of (II) are satisfied. But the solutions are $x=ce^t$, and these solutions are all unrestricted but fail to satisfy (6) when $c\neq 0$.

In order to prove (II*), it will first be shown that both inequalities (9) hold for every unrestricted solution $x=x(t)$ of (2). As will be clear from the proof, it will be sufficient to verify the first of the inequalities (9); in other words, there cannot exist a pair of positive numbers a, T satisfying (12). Suppose, if possible, that there does exist such a pair a, T. Then, since $a>0$, it is seen from (19), (11) and (10) that

$$g(x(t)) \geqq g^*(a) > 0 \qquad \text{if } t > T. \tag{29}$$

Let $x=x_1(t)$, where $0\leqq t<\infty$, be a solution of (2) satisfying (6), that is,

$$x_1'(t) = g(x_1(t)) + f(t), \tag{30}$$

and

$$x_1(t) \to 0 \quad \text{as} \quad t \to \infty. \tag{31}$$

The existence of such an $x_1(t)$ is assured by (II). Then it is clear from (31), $g(0)=0$, and the continuity of $g(x)$ at $x=0$ that, if T is sufficiently large,

$$| g(x_1(t)) | < g^*(a)/2 \qquad \text{if } t > T, \tag{32}$$

and

$$| x_1(t) | < a/2 \qquad \text{if } t > T. \tag{33}$$

By (2) and (30),

$$x'(t) - x_1'(t) = g(x(t)) - g(x_1(t)), \tag{34}$$

while (29) and (32) show that $g(x)-g(x_1)>g^*(a)/2>0$ if $t>T$. Thus, $x'-x_1'>0$ if $t>T$; so that $x-x_1$ is monotone increasing on this half-line.

The relations (12) and (33) imply that $x-x_1>a/2$ if $t>T$; so that, by (10) and (11),

$$g(x - x_1) \geqq g^*(a/2) > 0 \qquad \text{if } t > T.$$

Hence, it follows from (31) and the uniform continuity of $1/g(x)$ on any closed bounded interval (not containing $x=0$) that the ratio

$$(g(x) - g(x_1))/g(x - x_1) \tag{35}$$

tends to 1, as $t\to\infty$, if $x=x(t)$ remains bounded. On the other hand, if $x(t)$ is not bounded, the monotony of the function $x(t)-x_1(t)$ and (31) imply that $x(t)\to\infty$, as $t\to\infty$. Consequently, (31) and the assumption on the ratio (28) mean that the function (35) is bounded from below by a positive constant c for large values of t. (Actually, the assumption on (28) involves boundedness from above, but it is clear that boundedness from above for all large $|x|$ and small $|x_1|$ implies that the ratio cannot come close to 0.) Thus, whether $x(t)$ is or is not bounded, there exists a positive constant c such that

$$(g(x) - g(x_1))/g(x - x_1) > c \qquad \text{if } t > T,$$

provided that T is chosen sufficiently large.

Hence, (34) and a quadrature show that

$$\int_T^{T+v} (x' - x_1')/g(x - x_1)dt > cv \qquad (v > 0).$$

Since $x(t) - x_1(t)$ is monotone, the change of variables

$$u = u(t) = x(t) - x_1(t)$$

transforms the last inequality into

$$\int_{u_1}^{u_2} du/g(u) > cv,$$

where $u_1 = u(T)$ and $u_2 = u(T+v)$. If $v \rightarrow \infty$, the assumption (27) is contradicted (since $u_1 = x(T) - x_1(T) \geqq a/2 > 0$). This contradiction shows that the inequalities (9) must hold.

In order to complete the proof of (6), it will be shown that both inequalities (15) hold. Again, it will be sufficient to verify only one of them, say, the second one. Suppose, if possible, that the latter is false. Then there exists a positive constant b such that $x(t)$ exceeds b for certain, arbitrarily large, values of t. On the other hand, the first of the inequalities (9) implies that $b/2$ exceeds $x(t)$ for some other, arbitrarily large, values of t. Hence, by the continuity of $x(t)$, there exist arbitrarily large values of u and certain, corresponding, values $v = v_u > 0$ satisfying

$$x(u) = b, \qquad x(u + v) = b/2$$

and

$$x(t) > b/2 \quad \text{if} \quad u < t < u + v.$$

The remainder of the proof of (II*) is similar to the end of the proof of (I) and will, therefore, be omitted.

10. When the "smallness" condition (5) on $f(t)$ is strengthened, the corresponding assertion (6) concerning the "smallness" of the solutions $x = x(t)$ can be refined.

(III) *Let $g(x)$ and $f(t)$ be continuous on the line $-\infty < x < \infty$ and on the half-line $0 \leqq t < \infty$, respectively. Suppose that $f(t)$ is of class (L^p) for some $p \geqq 1$,*

$$\int_0^\infty |f(t)|^p dt < \infty. \tag{36}$$

Then (5) *holds* (*so that the assertions of* (I), (II) *or* (II*) *are valid if* $g(x)$ *satisfies the corresponding hypotheses*). *Also, if* (2) *possesses for* $T \leqq t < \infty$ *a solution* $x = x(t)$ *satisfying* (6), *then* $g(x(t))$ *and* $x'(t)$ *are of class* (L^p),

$$\int_T^\infty |g(x(t))|^p dt < \infty \quad \text{and} \quad \int_T^\infty |x'(t)|^p dt < \infty. \tag{37}$$

The Hölder inequality gives

$$\left|\int_u^{u+v} f(t)dt\right| < v^{1/q}\left(\int_u^{u+v} |f(t)|^p dt\right)^{1/p},$$

where $1/q + 1/p = 1$ (and $1/q = 0$ if $p = 1$). If $v > 0$, then $v^{1/q} < 1 + v$, so that

$$\left|\int_u^{u+v} f(t)dt\right| \Big/ (1 + v) < \left(\int_u^\infty |f(t)|^p\right)^{1/p}.$$

Hence, (5) follows from (36).

In order to prove the assertion concerning (37), let $\delta(x)$ be defined as $-1, 0$ or $+1$ according as $g(x)$ is negative, zero or positive. Then, by (2),

$$|g(x(t))|^p = \delta(x(t))\,|g(x(t))|^{p-1}(x'(t) - f(t)).$$

But, by (6), the integral

$$\int_T^t \delta(x(s))\,|g(x(s))|^{p-1}x'(s)ds = \int_{x(T)}^{x(t)} \delta(s)\,|g(s)|^{p-1}ds$$

tends to

$$\int_{x(T)}^0 \delta(s)g(s)ds$$

and, hence, is $O(1)$, as $t \to \infty$. Thus

$$\int_T^t |g(x(s))|^p ds < \int_T^t |g(x(s))|^{p-1}|f(s)|\,ds + O(1).$$

If $p = 1$, it follows from (36) that $g(x(t))$ is of class $(L^p) = (L^1)$. Hence, it may be supposed that $p > 1$. Then Hölder's inequality implies that the integral on the right of the last inequality is majorized by

$$\left(\int_T^t |g(x(s))|^p ds\right)^{1/q}\left(\int_T^t |f(s)|^p\right)^{1/p}$$

where $1/p + 1/q = 1$. It then follows from (36) that

$$\int_T^t |g(x(s))|^p ds < O(1)\left(\int_T^t |g(x(s))|^p ds\right)^{1/q} + O(1),$$

or, since $q>1$,

$$\int_T^t |g(x(s))|^p ds = O(1).$$

This proves the first assertion (37). The second assertion follows from the first, from (2) and from the fact that the class (L^p) is linear.

This completes the proof of (III).

PART II

11. The theorems just proved will be applied in order to obtain the asymptotic behavior of the solutions of the linear differential equation

$$y'' = \phi(t)y \tag{38}$$

in the "hyperbolic" case, when $\phi(t)$ is "nearly" a positive constant for large t. It can be supposed that this constant is 1 (otherwise the unit of length on the t-axis can be changed), then the difference $f(t)=\phi(t)-1$ is "small" and (38) becomes

$$y'' = (1+f(t))y. \tag{39}$$

An application of (I) or (II) leads to

(IV) *Let $f(t)$, where $0\leqq t<\infty$, be a continuous function. Then $f(t)$ satisfies* (5) *if and only if the linear differential equation* (39) *possesses a solution $y=y(t)$ with the property that*

$$\text{either } y'(t)/y(t)\to 1 \quad \text{or} \quad y'(t)/y(t)\to -1 \qquad (t\to\infty) \tag{40}$$

(which implies that $y(t)$ cannot vanish for large t).

A classical result of Poincaré is that, under conditions of analyticity, (5′) is sufficient for the existence of solutions satisfying (40); Perron [3, pp. 158–160] has shown that the assumptions of analyticity can be omitted. The condition (5′) makes one obvious simplification; namely, that $1+f(t)>0$ for large t, so that the Sturm comparison theorem implies that a nontrivial solution of (39) does not vanish for large t. However, in the present case, the Sturm theorem is not available, since (5), or even (5″), does not imply that $1+f(t)$ becomes positive for large t.

12. The sufficiency of (5) will first be proved. An introduction of the new dependent variable,

$$x = y'/y \tag{41}$$

transforms (39) into

$$x' = 1 - x^2 + f(t). \tag{42}$$

If $y=y(t)$ is a solution of (39) and if $y(t)\neq 0$ on some t-interval, then (41) defines a solution of (42) on the same t-interval. Conversely, if $x=x(t)$ is a solution of (42) on some t-interval, then a quadrature of (41) gives a function $y=y(t)$ which does not vanish and which is a solution of (39) on the same t-interval.

The differential equation (42) is of the type (2), where $g(x)=1-x^2$. It is obviously possible to define a continuous function $g_1(x)$, for $-\infty<x<\infty$, satisfying (4), the identity

$$g_1(x) = 1 - x^2 \quad \text{if} \quad x > 0, \tag{43}$$

and the inequalities

$$g_1(x) < 0 \quad \text{if} \quad 1 < x < \infty \quad \text{and} \quad g_1(x) > 0 \quad \text{if} \quad -\infty < x < 1.$$

Then the conditions imposed by Theorem (I) on $g(x)$ and $f(t)$ are satisfied by the functions $g_1(x)$ and $f(t)$, respectively, with the modification that the zero of $g_1(x)$ occurs at $x=1$ instead of $x=0$.

It is clear that (I) implies that all solutions $x=x(t)$ of the differential equation

$$x' = g_1(x) + f(t) \tag{44}$$

are unrestricted and satisfy

$$x(t) \to 1 \quad \text{as} \quad t \to \infty. \tag{45}$$

In particular, $x(t)>0$ for sufficiently large t, say $t>T$. But then $x(t)$ is also a solution of (42) for $t>T$, in view of (43) and (44). Consequently, by the remark following (42), the differential equation (39) possesses solutions which do not vanish for large t. Also, by (41) and (45), these solutions satisfy the first alternative of (40).

It is clear that, if use is made of theorem (II) rather than (I), the existence of solutions satisfying the second alternative of (40) can be established.

13. As to the proof of the necessity of condition (5) in (IV), it will be clear from the proof (and/or the proof of (V) below) that it is sufficient to consider the case that (39) possesses a solution $y=y(t)$ which does not vanish for large t and satisfies the first alternative in (40).

In this case, (41) defines for large t a function $x=x(t)$ which satisfies the limit relation (45) and the differential equation (42). This assures that $f(t)$ satisfies (5); cf. the remarks on the necessity of (5) in (I) in §2.

14. In order to avoid an interruption to the proofs below, the following corollary of (IV) will first be established:

(V) *If* $f(t)$, *where* $0\leqq t<\infty$, *is a continuous function satisfying* (5), *then every solution* $y=y(t)\not\equiv 0$ *of the differential equation* (39) *has at most a finite number of zeros and satisfies* (40); *furthermore, every solution* $y=y(t)$ *is the superposition*

$$y = c_1 y_1 + c_2 y_2 \tag{46}$$

of two particular solutions $y=y_1(t)$ *and* $y=y_2(t)$, *which satisfy the asymptotic relations*

$$y_1'(t)/y_1(t) \to 1, \tag{47}$$

$$y_2'(t)/y_2(t) \to -1 \tag{48}$$

and

$$y_1(t)y_2(t) \to 1, \tag{49}$$

as $t\to\infty$.

By (IV), the condition (5) implies the existence of solutions $y=y(t)$ of (39) which do not vanish for large t. Then the separation property of zeros of the solutions of differential equations (39) imply that all nontrivial solutions do not vanish for large t. Since the zeros of a solution cannot cluster at a finite t-value, every nontrivial solution of (39) has at most a finite number of zeros on $0\leqq t<\infty$.

Also, (IV) implies the existence of solutions $y=y_1(t)$ and $y=y_2(t)$ of (39) satisfying (47) and (48), respectively. Since (47) and (48) imply $y_1(t)\to\infty$ and $y_2(t)\to 0$, as $t\to\infty$, these solutions are linearly independent and so every solution $y=y(t)$ is a linear combination, (46), of them. If this solution (46) is not identically zero, then the first or second alternative of (40) holds according as $c_1\neq 0$ or $c_1=0$.

15. It remains to be verified that y_1 and y_2 can be chosen so as to fulfill (49), as well as (47) and (48). If $y=y_1(t)$ and $y=y_2(t)$ are any two solutions of (39), then their Wronskian relation is

$$y_1'(t)y_2(t) - y_1(t)y_2'(t) = c,$$

where c is a constant. If $y=y_1(t)$ is a function which satisfies (47), then for any number θ, $0<\theta<1$, and for sufficiently large t,

$$|y_1(t)| > \text{const. } e^{\theta t}$$

and so

$$\int^\infty dt/(y_1(t))^2 < \infty. \tag{50}$$

A quadrature of the Wronskian relation gives

$$y_2(t) = y_1(t)\left(C - c\int_t^\infty ds/(y_1(s))^2\right),$$

where C is another constant. Conversely, if $y_1(t)$ is any solution of (39) satisfying (50), then the last formula line defines, for arbitrary constants C and c, a solution $y=y_2(t)$ of (39). Particularize the constants by placing $C=0$ and $c=-2$, so that

$$y_2(t) = 2y_1(t)\int_t^\infty ds/(y_1(s))^2.$$

Since $y_1(t)\to\infty$, it follows from l'Hospital's rule that $y_1(t)y_2(t)$ has a limit, which is equal to the limit of the ratio

$$2\left(\int_t^\infty ds/y_1^2(s)\right)' : (1/y_1^2(t))',$$

provided that this ratio has a limit. But this ratio is $y_1(t)/y_1'(t)$. Since $y=y_1(t)$ satisfies (47), this proves (49). On dividing the Wronskian relation (with $c=-2$) by $y_1(t)y_2(t)$, it is seen that (48) is satisfied by $y=y_2(t)$. This completes the proof of (V).

16. From the point of view of an asymptotic integration of (39), the result (40), that $y'\sim\pm y$, is usually insufficient. The first step in the improvement of this result to an asymptotic formula for $y(t)$ is:

(VI) *If* $f(t)$, *where* $0\leqq t<\infty$, *is a continuous function satisfying* (5), *then*

$$\lim_{t\to\infty} y(t)\exp\left(-t-2^{-1}\int_0^t f(s)ds\right) \tag{51}$$

exists as a finite limit for every solution $y=y(t)$ *of* (39).

The existence of the limit (51) need not imply an asymptotic formula for the solution $y(t)$, inasmuch as this limit may be 0 for every solution $y=y(t)$. This possibility is illustrated by the following example: It is easily verified that if $y=y_1(t)=\exp(t+2t^{1/2})$, then $y_1'=(1+t^{-1/2})y_1$, so that $y=y_1(t)$ is a solution of the differential equation (39), where

$$f(t) = 2t^{-1/2}+t^{-1}-t^{-3/2}/2 \qquad (t>1)$$

and (47) is satisfied. But, as $t\to\infty$,

$$y_1(t)\exp\left(-t-2^{-1}\int_1^t f(s)ds\right) = O(1)t^{-1/2},$$

so that the corresponding limit (51) exists and is zero. On the other hand, since, in this case, (39) has a solution $y=y_1(t)$ satisfying (47), it also has a (linearly independent) solution $y=y_2(t)$ satisfying (48). But this latter solution tends to 0 as $t\to\infty$, and so the corresponding limit (51) certainly exists and is zero. Since an arbitrary solution is a linear combination of the two solutions, y_1 and y_2, it follows that (51) exists and is 0 for all solutions, $y=y(t)$.

17. In order to prove (VI), let $y=y(t)\not\equiv 0$ be any solution of (39). Then

$y(t)$ cannot vanish for large t and satisfies (40), by Theorem (V). Let $x=x(t)$ be defined, for large t, as

$$x = -1 + y'/y, \tag{52}$$

so that (40) implies

$$x(t) \to 0 \quad \text{or} \quad x(t) \to -2 \qquad (t \to \infty). \tag{53}$$

Since $y(t)$ is a solution of (39), it follows from (52) that $x=x(t)$ satisfies

$$x' + x^2 + 2x = f(t). \tag{54}$$

A quadrature of (52) gives

$$y(t) = \text{const.} \exp\left(t + \int^t x(s)ds\right),$$

and so

$$\begin{aligned} y(t) \exp&\left(-t - 2^{-1}\int^t f(s)ds\right) \\ &= \text{const.} \exp\left(\int^t x(s)ds - 2^{-1}\int^t f(s)ds\right). \end{aligned} \tag{55}$$

Hence, in order to verify that the limit (51) exists as a finite limit, it is sufficient to show that

$$2\int^t x(s)ds - \int^t f(s)ds \tag{56}$$

tends either to a finite limit or to $-\infty$. But (54) shows that the expression (56) is equal to

$$-x(t) - \int^t (x(s))^2 ds.$$

Thus, by (53), the function (56) tends to the limit

$$\text{const.} - \int^\infty (x(s))^2 ds,$$

which may be $-\infty$. This completes the proof of (VI).

Remark. It can be mentioned for use below that the limit (51) is different from zero if and only if the function $x=x(t)$, defined by (52), is of class (L^2) (on some half-line),

$$\int^\infty (x(t))^2 dt < \infty. \tag{57}$$

This is clear from the preceding deduction.

18. A slight refinement of the conditions on $f(t)$ in (VI) leads to an asymptotic formula.

(VII) *If $f(t)$, where $0 \leqq t < \infty$, is a continuous function satisfying*

$$\int_0^\infty |f(t)|^p dt < \infty \text{ for some } p, \qquad 1 \leqq p \leqq 2, \tag{58}$$

then every solution $y = y(t)$ of (39) *is the superposition* (46) *of two particular solutions $y = y_1(t)$ and $y = y_2(t)$, which satisfy the asymptotic relations*

$$y_1(t) \sim \exp\left(t + 2^{-1}\int_0^t f(s)ds\right), \qquad y_1'(t) \sim y_1(t) \tag{59}$$

and

$$y_2(t) \sim \exp\left(-t - 2^{-1}\int_0^t f(s)ds\right), \qquad -y_2'(t) \sim y_2(t), \tag{60}$$

as $t \to \infty$.

That (VII) can be false when $f(t)$ is of class (L^p) for every $p > 2$ is shown by the example in §16.

In the particular case $p = 1$, the theorem (VII) is contained in a result of Bôcher [1, pp. 41–48]; for a simple proof, cf. Wintner [7]. In the case $p = 2$, (VII) has been proved by Wintner [7] under the additional restriction that $f(t)$ has a finite total variation on $0 \leqq t < \infty$. Actually, Wintner has shown that if $f(t)$ has a finite total variation on $0 \leqq t < \infty$ and satisfies (5′), then (39) has two linearly independent solutions $y = y_1(t)$ and $y = y_2(t)$ satisfying

$$y_1(t) \sim \exp\left(\int^t (1 + f(s))^{1/2} ds\right), \qquad y_1'(t) \sim y_1(t) \tag{59 bis}$$

and

$$y_2(t) \sim \exp\left(-\int^t (1 + f(s))^{1/2} ds\right), \qquad -y_2'(t) \sim y_2(t), \tag{60 bis}$$

respectively; so that, if $f(t)$ is also of class (L^2), these functions y_1 and y_2 are asymptotically proportional to functions satisfying (59) and (60), respectively. Since (VII) shows that the condition of bounded variation is superfluous in order to assure the existence of solutions satisfying (59), (60), the question arises as to whether or not that condition is also superfluous in order to assure the existence of solutions satisfying (59 bis) and (60 bis). However, it will be shown in §23 that if $f(t)$ satisfies (5′), (5″) and is of class (L^p) for every $p > 2$, then (39) need not have solutions satisfying either (59 bis) or (60 bis).

When the equation (39) is of the "elliptic" type, say

$$y'' + (1 + f(t))y = 0 \tag{39 bis}$$

(where $f(t)$ is "small"), it has been shown by Wintner [5, pp. 252–253], that if $f(t)$ is of bounded variation on $0 \leqq t < \infty$ and of class (L^2), then (39 bis) possesses two solutions $y = y_1(t)$ and $y = y_2(t)$ satisfying

$$y_1(t) - \cos\left(t + 2^{-1}\int_0^t f(s)ds\right) \to 0,$$

and

$$y_2(t) - \sin\left(t + 2^{-1}\int_0^t f(s)ds\right) \to 0,$$

as $t \to \infty$. Since these are the analogues of (59) and (60), the question also arises here as to whether or not the condition of bounded variation is superfluous. The answer is that the bounded variation condition cannot be omitted, cf. Wintner [5, pp. 255–256], so that (VII) has no analogue for the "elliptic" case (39 bis). This is one of the few instances where asymptotic integration results for the "hyperbolic" and "elliptic" cases are not completely parallel.

19. According to (III), the condition (58) implies (5). Hence, the assumptions of (VI) are fulfilled by the function $f(t)$ in (VII).

Let $y = y(t)$ be a solution of (39) satisfying the first alternative of (40). It will be shown that (58) implies that the limit (51) is not 0; hence, a constant multiple of this $y = y(t)$ is a solution $y = y_1(t)$ of (39) satisfying (59). In terms of $y(t)$, define a function $x = x(t)$, for large t, by (52). In virtue of the remark preceding (VII), it is sufficient to prove that (57) holds.

The function $x = x(t)$ is, for large t, a solution of the differential equation (54), which is of the type (2), where $g(x) = -x^2 - 2x$. Also, since $y = y(t)$ satisfies the first alternative in (40), $x = x(t)$ satisfies the first alternative in (53), that is, (6). Hence, (58) implies that Theorem (III) is applicable. Consequently, $-g(x(t)) = x^2(t) + 2x(t)$ is of class (L^p). Since $p \leqq 2$, $x^2(t) + 2x(t)$ is of class (L^2), in virtue of (6). The limit relation (6) also shows that for large t,

$$(x^2(t) + 2x(t))^2 \geqq x^2(t),$$

and so (57) holds.

This completes the proof of the existence of a solution $y = y_1(t)$ satisfying (59). The existence of a linearly independent solution $y = y_2(t)$ of (39) satisfying (60) is a consequence of (V). This completes the proof of (VII).

20. The theorem (VII) implies the following corollary:

(VIII) *If* $f(t)$, $0 \leqq t < \infty$, *is a continuous function satisfying* (5″) *and* (58), *then every solution* $y = y(t)$ *of* (39) *is the superposition of two particular solutions* $y = y_1(t)$ *and* $y = y_2(t)$, *which satisfy the asymptotic relations*

(61) $$y_1(t) \sim e^t, \qquad y_1'(t) \sim e^t$$

and

(62) $$y_2(t) \sim e^{-t}, \qquad -y_2'(t) \sim e^{-t},$$

as $t \to \infty$.

This follows at once from (VII), since (5″) implies that $\exp\left(t+2^{-1}\int_0^t f(s)ds\right) \sim \text{const. } e^t$, as $t \to \infty$, where const.$\neq 0$. (Of course, when $p=1$ in (58), the hypothesis (5″) is redundant.)

Neither of the conditions (5″), (58) can be omitted in (VIII). In fact, it is clear from (VII) that if $f(t)$ satisfies (58), but not (5″), then the assertions of (VIII) cannot hold. It is also true, but not obvious, that if $f(t)$ satisfies (5″) but not (58), then the assertions of (VIII) can be false. The theorem (VI) shows that when (5″) is satisfied and the assertion of (VIII) are false, then $y(t)=o(e^t)$, as $t \to \infty$, holds for all solutions $y=y(t)$ of (39). Thus the statement that condition (58) cannot be omitted in (VIII) is contained in

(IX) *There exist continuous functions* $f(t)$, *where* $0 \leqq t < \infty$, *satisfying* (5′) *and* (5″), *of class* (L^p) *for every* $p>2$, *and having the property that*

(63) $$y(t)e^{-t} \to 0 \quad as \quad t \to \infty$$

holds for every solution $y=y(t)$ *of* (39).

21. Let $x=x(t)$, where $0 \leqq t < \infty$, possess a continuous derivative and satisfy

(64) $$x(t) \to 0 \quad \text{as} \quad t \to \infty.$$

Define $y=y(t)$ for $0 \leqq t < \infty$ by the formula

(65) $$y(t) = \exp\left(t + \int_0^t x(s)ds\right),$$

so that (52) holds at every $t \geqq 0$. If $f(t)$, where $0 \leqq t < \infty$, is defined in terms of $x(t)$ and $x'(t)$ by (54), then (52) implies that (65) is a solution of (39). Also (52) and (64) show that (65) satisfies the first alternative of (40).

In order to prove (IX), it is sufficient to choose the function $x(t)$ so that

(66) $$\int_0^t x(s)ds \to -\infty \quad \text{as} \quad t \to \infty$$

holds and, at the same time, $f(t)$ is of class (L^p) for every $p>2$ and satisfies (5′) and (5″). For (66) means that (63) holds for the function (65). On the other hand, theorem (V) (which is applicable, since (5′) implies (5)) assures the existence of a solution, linearly independent of (65), which satisfies the second alternative of (40) and, hence, tends to 0 as $t\to\infty$. Thus (63) will hold for this latter solution and, hence, for all solutions $y=y(t)$ of (39).

If $x(t)$ is chosen so that

$$x'(t)\to 0 \quad \text{as} \quad t\to\infty, \tag{67}$$

then (54) and (64) assure (5′). It also follows from (64) and (54) that in order to assure (5″), it is sufficient that $x(t)$ have the property that

$$\lim_{t\to\infty}\int_0^t (x^2(s)+2x(s))ds \text{ exists} \tag{68}$$

as a finite limit. Finally, by (54), $f(t)$ is of class (L^p) for every $p>2$ if $x(t)$ and its derivative, $x'(t)$, are each $O(t^{-1/2})$ as $t\to\infty$; these estimates, of course, imply (64) and (67) as well.

To summarize, if the existence of a function $x(t)$ possessing a continuous derivative and satisfying $x(t)=O(t^{-1/2})$, $x'(t)=O(t^{-1/2})$, (66) and (68) is established, then the proof of (IX) will be complete.

22. It will be shown that the above requirements are satisfied if the function $x(t)$ is defined as follows:

$$x(t)=\begin{cases} -\,t^{-1/2}\sin^2 t & \text{if } 2n\pi\leqq t<(2n+1)\pi \\ (t-\pi)^{-1/2}(1-(t-\pi)^{-1/2}\sin^2 t)\sin^2 t & \text{if } (2n+1)\pi\leqq t<(2n+2)\pi, \end{cases} \tag{69}$$

where $n=1,2,\cdots$ (and $x(t)=0$ if $0\leqq t<2\pi$).

Since $\sin^2 t$ vanishes in the second order at $t=2\pi, 3\pi, \cdots$, this $x(t)$ has a continuous derivative. It is also seen from (69) that $x(t)=O(t^{-1/2})$ as $t\to\infty$, and the differentiation of (69) shows that $x'(t)$, too, is $O(t^{-1/2})$.

The remaining conditions are (66) and (68). But it is seen from (69) that

$$\int_{2n\pi}^{(2n+2)\pi} x(t)dt = -\int_{2n\pi}^{(2n+1)\pi} t^{-1}\sin^4 t\,dt. \tag{70}$$

Since, as $n\to\infty$, the expression on the right of (70) is asymptotically equal to

$$-\int_{2n\pi}^{(2n+1)\pi}\sin^4 t\,dt/n = -\text{ const.}/n, \quad \text{where const.} > 0,$$

(66) follows from (64) and (70).

Finally, it is seen from (69) that

$$\int_{2n\pi}^{(2n+2)\pi} x^2(t)dt = \int_{2n\pi}^{(2n+1)\pi}(1+1-2t^{-1/2}\sin^2 t+t^{-1}\sin^4 t)t^{-1}\sin^4 t\,dt,$$

which implies that, as $n\to\infty$,

$$(71)\qquad \int_{2n\pi}^{(2n+2)\pi} x^2(t)dt = 2\int_{2n\pi}^{(2n+1)\pi} t^{-1}\sin^4 t\, dt + O\left(\int_{2n\pi}^{(2n+1)\pi} t^{-3/2}dt\right).$$

Since the O-term of (71) is $O(n^{-3/2})$, it follows from (70) and (71) that

$$(72)\qquad \int_{2n\pi}^{(2n+2)\pi} (x^2(t) + 2x(t))dt = O(n^{-3/2}).$$

Since (72) and (64) imply (68), the proof of (IX) is complete.

23. It will be shown that even though the function $f(t)$ defined by (54) and (69) satisfies (5′), (5″) and is of class (L^p) for every $p>2$, the differential equation (39) has no solution satisfying (59 bis). This will furnish the proof of the statement, §18, concerning the impossibility of the omission of the condition of bounded variation in the theorem of Wintner [7] mentioned in connection with (59 bis) and (60 bis). Since (65) is a solution of (39), it is sufficient to verify that the difference

$$(73)\qquad \int^t (1 + x(s))ds - \int^t (1 + f(s))^{1/2}ds$$

does not tend to a finite limit. But, by (5′),

$$(1 + f)^{1/2} = 1 + f/2 - f^2/8 + O(|f|^3) \qquad (t\to\infty),$$

so that, by (54),

$$(1 + f)^{1/2} = 1 + x + x'/2 - x'^2/8 - x'x^2/4 - x'x/2 - x^3/8 + O(|f|^3).$$

Hence, the difference (73) equals

$$-\, x(t)/2 + \int^t x'^2(s)ds/8 + x^3(t)/12 + x^2(t) + \int^t (x^3(s)/8 + O(|f|^3))ds.$$

Since $x(t)=O(t^{-1/2})$ and $x'(t)=O(t^{-1/2})$, it follows from (54) that $f(t)=O(t^{-1/2})$. Thus, by (64), the difference (73) tends to a finite limit if and only if

$$\int^t x'^2(s)ds$$

does. But it is clear from (69) that $x'(t)$ is not of class (L^2). Hence, (73) does not tend to a finite limit and so (39) does not possess a solution $y=y_1(t)$ satisfying (59 bis). It cannot have a solution $y=y_2(t)$ satisfying (60 bis) either, by virtue of (V).

24. The proof of theorem (IX) in §§21–22 is a special case of a general method for the construction of counter-examples in the theory of the linear differential equation (39). The main point of this construction is the fact that the coefficient function $f(t)$ in (39) and a corresponding solution (65) of (39) are defined in terms of a single function $x(t)$. The connection between

$x(t)$ and $f(t)$ is a first order differential equation, namely, the Ricatti equation (55). This method is an analogue of the procedure used by Wintner [5, pp. 268–269], in the "elliptic" case.

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